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Phase transition in a difference equation model of traffic flow

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Abstract. A difference equation is presented to describe traffic flow on a highway. The difference equation model is derived from the optimal velocity models formulated in terms of the differential equations. It is compared with the differential equation models. We investigate phase transitions among the freely moving phase, the coexisting phase in which jams appear, and the uniform congested phase. The linear stability theory is applied and the neutral stability line is obtained. We find the critical point below which no jams appear. To derive the modified Korteweg–de Vries equation near the critical point we apply the reductive perturbation method. We also compare the nonlinear analysis result with that of the optimal velocity model. It is shown that the critical point and the amplitude of the jam are different from those of the optimal velocity model.

1. Introduction

Recently, traffic problems have attracted considerable attention [1–23]. To know the behaviour of traffic is important in our daily life. Various traffic models have been proposed and studied: car following models [2–9], cellular automaton models [10–15], gas kinetic models [16–21] and fluid dynamic models [22, 23]. These models are successfully applied to the computer simulation of traffic. Kerner and Rehborn [24, 25] have observed the phase transitions between the freely moving phase and the coexisting phase in actual traffic.

The fluid dynamic model and the optimal velocity model have been studied analytically using the linear stability theory and the nonlinear analysis method [26, 27]. Kurtze and Hong [26] have derived the Korteweg–de Vries (KdV) equation from the fluid dynamic traffic model. Komatsu and Sasa [27] have derived the modified KdV equation from the optimal velocity model giving traffic jams in terms of a kink density wave. For the nonlinear analysis, it is important that there exists a critical point. Only near the critical point, is it possible to apply the reductive perturbation method to the optimal velocity model. Until now, the nonlinear wave equation could not successfully be derived from the cellular automaton model. This may be due to the nonanalytical properties of the cellular automaton model.

The fluid dynamical model is described in terms of the partial differential equations. The optimal velocity model is also described by the differential equation of motion. To date, a traffic model described by the difference equation is unknown. There are also few applications of the nonlinear analysis to such a difference equation system.

In this paper, we propose a traffic model described by the difference equation. The traffic model is derived from the optimal velocity models. We study the traffic behaviour

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by using the linear stability theory, nonlinear wave analysis and computer simulation. We show that there is a critical point below which no phase transition occurs. We also derive the modified KdV equation in order to describe the traffic jams. Finally we compare our analytical results with the simulation.

2. Difference equation model

We derive the difference equation from the optimal velocity models formulated in terms of differential equations. To date, the two optimal velocity models have been proposed independently by Newell [28] and Bando *et al* [3]. The optimal velocity model proposed by Newell is given by the equation of motion of car j

$$\mathrm{d}x_j(t+\tau)/\mathrm{d}t = V(\Delta x_j(t)) \tag{1}$$

where $\Delta x_j (= x_{j+1} - x_j)$ is the headway and τ is the delay time. The idea is that a driver adjusts the car velocity $\mathrm{d}x_j/\mathrm{d}t$ according to the observed headway $\Delta x_j(t)$. The delay time τ allows for the time lag that it takes for the car velocity to reach the optimal velocity $V(\Delta x_j)$ when the flow is varying.

Bando *et al* [3] proposed an optimal velocity model described by the following differential equation

$$\mathrm{d}^2x_j/\mathrm{d}t^2 = a(V(\Delta x_j(t)) - \mathrm{d}x_j/\mathrm{d}t) \tag{2}$$

where a is the sensitivity. In this model, the inverse of sensitivity a corresponds to the delay time τ in equation (1). Car j is controlled in such a way that the car velocity $\mathrm{d}x_j/\mathrm{d}t$ adjusts the optimal velocity $V(\Delta x_j)$ depending upon the headway Δx_j. Unfortunately, Bando's model cannot be compared with Newell's model [28, 29] since the optimal velocity functions are different for the two models.

We wish to derive the difference equation from Newell's optimal velocity model. We rewrite equation (1) to obtain the difference equation, $x_j(t+\tau+\Delta t) - x_j(t+\tau) = V(\Delta x_j(t))\Delta t$ where Δt is an infinitesimal value of time. Setting $\Delta t = \tau$, we obtain the following difference equation

$$x_j(t+2\tau) = x_j(t+\tau) + V(\Delta x_j(t))\tau. \tag{3}$$

Similarly, we derive the difference equation (3) from Bando's optimal velocity model. By rewriting equation (2), we obtain the difference equation, $\{x_j(t+2\Delta t) - 2x_j(t+\Delta t) + x_j(t)\}/(\Delta t)^2 = a\{V(\Delta x_j) - (x_j(t+\Delta t) - x_j(t))/\Delta t\}$. By replacing $\Delta t = \tau$ and $a = 1/\tau$, we obtain equation (3).

In the difference equation model described by equation (3), the position of car j at time $t+2\tau$ is determined by the position at time $t+\tau$ and the headway at time t. The difference equation model will be more suitable to computer simulation than the differential equation models.

We consider the optimal velocity function. When the headway is less than the safety distance, the car velocity is reduced and small enough to prevent it from crashing into the preceding car. On the other hand, if the headway is larger than the safety distance, the car moves with higher velocity. Also, the car does not exceed the maximum velocity. Thus, the optimal velocity is a function having the following properties: a monotonically increasing function with an upper bound (maximal velocity). Here, we choose the same optimal velocity function as that used by Bando *et al* [3].

$$V(\Delta x_j) = \tanh(\Delta x_j - h_c) + \tanh(h_c) \tag{4}$$

where h_c is the safety distance and the optimal velocity function has a turning point (inflection point) at $\Delta x_j = h_c$: $V''(h_c) = 0$. It is important that the optimal velocity function has a turning point, otherwise, we cannot derive the modified KdV equation giving traffic jams in terms of a kink density wave.

Newell and Whitham [28, 29] have used the different optimal velocity function: $V(\Delta x_j) = v\{1 - \exp[-(\gamma/v)(\Delta x_j - L)]\}$. This optimal velocity function does not have a turning point. We note that if we choose Newell's function as the optimal velocity, we cannot derive the modified KdV equation from the difference equation model.

3. Linear stability theory

We apply the linear stability method to the traffic model described by the difference equation (3). The linear stability method is similar to that used by Bando *et al* [3]. To do so, we initially consider the stability of the uniform traffic flow. The uniform traffic flow is defined by such a state that all cars move with identical headway h and optimal velocity $V(h)$. The solution $x_{j,0}(t)$ of the uniform steady state is given by

$$x_{j,0}(t) = hj + V(h)t \qquad \text{with } h = L/N \tag{5}$$

where N is the number of cars and h is the car spacing (identical headway). Let $y_j(t)$ be a small deviation from the steady-state flow $x_{j,0}(t)$: $x_j(t) = x_{j,0}(t) + y_j(t)$. Then the linearized equation is obtained as follows

$$y_j(t + 2\tau) = y_j(t + \tau) + V'(h)\tau\Delta y_j(t) \tag{6}$$

where $V'(h)$ is the derivative of the optimal velocity function at $\Delta x = h$.

By expanding $y_j \sim \exp(\mathrm{i}kj + zt)$, the following equation of z is derived

$$\mathrm{e}^{z\tau}(\mathrm{e}^{z\tau} - 1) = V'\tau(\mathrm{e}^{\mathrm{i}k} - 1). \tag{7}$$

By expanding $z\tau = (z\tau)_1(\mathrm{i}k) + (z\tau)_2(\mathrm{i}k)^2 + \cdots$, the first- and second-order terms are obtained

$$(z\tau)_1 = V'\tau \qquad \text{and} \qquad (z\tau)_2 = (V'\tau/2)(1 - 3V'\tau). \tag{8}$$

If $(z\tau)_2$ is a negative value, the uniform steady-state flow becomes unstable for long wavelengths. When $(z\tau)_2$ is a positive value, the uniform flow is stable. The neutral stability condition is given by

$$V'(h)\tau = \tfrac{1}{3}. \tag{9}$$

Figure 1 shows the plot of headway Δx against τ where $h_c = 4$. The broken curve represents the neutral stability line. We find that there is a critical point at $\Delta x = h_c$ and $\tau = \frac{1}{3}$ where h_c is the headway at the turning point of the optimal velocity function (4) and $V'(h_c) = 1$. ($V'(h_c)$ is the derivative at the turning point). For small disturbances with long wavelengths, the uniform traffic flow is unstable if

$$\tau > 1/(3V'(h)). \tag{10}$$

When a small disturbance is added to the uniform traffic flow (with an identical headway and the optimal velocity) satisfying the above condition, its uniform flow becomes unstable and in time traffic jams will be formed. On the other hand, when a small disturbance is added to the uniform traffic flow satisfying $\tau < 1/(3V'(h))$, its uniform flow is always stable and the traffic flow remains uniform.

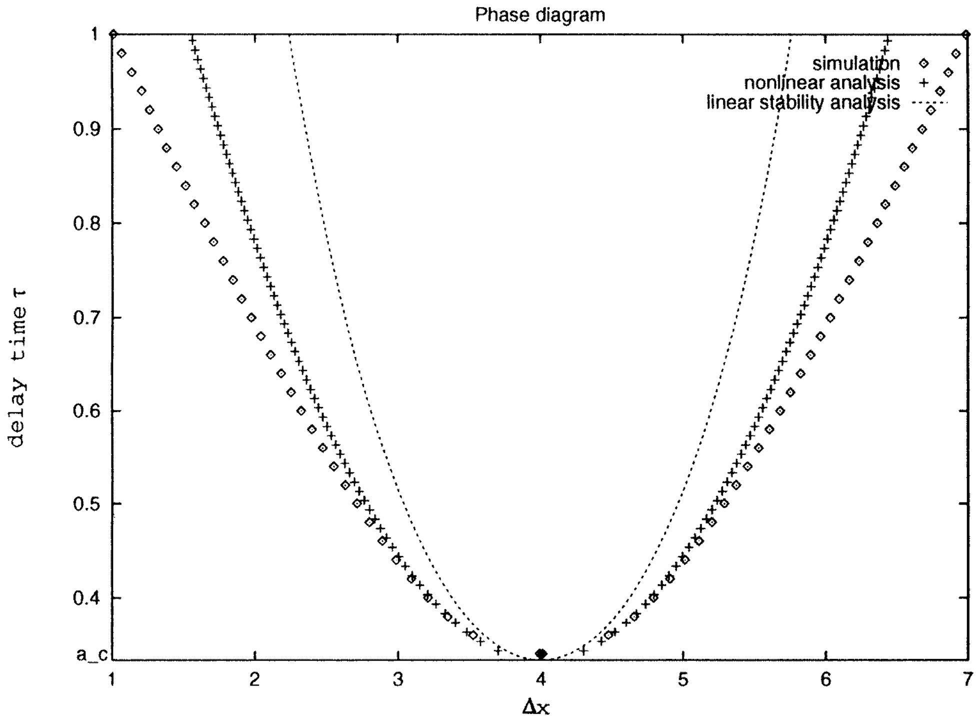

Figure 1. The phase diagram for the curve according to simulation results ($\diamond$ points), those according to nonlinear analysis ($+$ points) and linear stability analysis (broken curve), respectively. The safety distance h_c is chosen to be 4. Simulation and nonlinear analysis are in good agreement near the critical point. For the region between the coexisting curve obtained from nonlinear analysis and the neutral stability curve obtained from linear stability analysis, our model is in a metastable state.

4. Nonlinear analysis

We now consider a hydrodynamic mode in traffic flow on coarse-grained scales. The simplest way to describe the hydrodynamic mode is the long-wave expansion. We apply the reductive perturbation method [27, 30] to the difference equation (3). We rewrite equation (3) into the difference equation of headways

$$\Delta x_j(t+2\tau) = \Delta x_j(t+\tau) + V(\Delta x_{j+1}(t))\tau - V(\Delta x_j(t))\tau. \tag{11}$$

We consider the slowly varying behaviour at long wavelengths near the critical point. We wish to extract slow scales for the space variable j and time variable t. For $0 < \varepsilon \ll 1$, we therefore define the slow variables X and T:

$$X = \varepsilon(j + bt) \qquad \text{and} \qquad T = C_2\varepsilon^3 t. \tag{12}$$

We set the headway $\Delta x_j(t)$ as

$$\Delta x_j(t) = h_c + C_1\varepsilon r(X, T) \tag{13}$$

where constants C_1, C_2 and b are to be determined in order to simplify the coefficients of the nonlinear wave equation. By replacing equation (13) into equation (11) and expanding to the fifth-order of ε, we obtain the following nonlinear equation (see the appendix for details of the derivation)

$$\varepsilon^4\partial_T r = \varepsilon^2 C_{11}\partial_X r + \varepsilon^3 C_{12}\partial_X^2 r + \varepsilon^4 C_{13}\partial^3 r + \varepsilon^4 C_{31}\partial_X r^3 + \varepsilon^5[C_{14}\partial_X^4 r - 3b\tau\partial_T\partial_X r + C_{32}\partial_X^2 r^3] \tag{14}$$

where $X = (j + bt)\varepsilon$, $T = C_2\varepsilon^3 t$, $b = V'(h_c)$, $C_1 = (6C_2/|V'''(h_c)|)^{1/2}$, $C_2 = \frac{1}{27}$, $C_{11} = 0$, $C_{12} = \frac{27}{2}(1 - 3V'(h_c)\tau)$, $C_{13} = 1$, $C_{31} = -1$, $C_{14} = \frac{1}{2}$, $C_{32} = -\frac{1}{2}$.

We set the deviation from the critical point $(V'\tau)_c = \frac{1}{3}$ as

$$\varepsilon^2 = V'(h_c)\tau - \tfrac{1}{3}. \tag{15}$$

Thus we obtain the nonlinear wave equation

$$\partial_T r = \partial_X^3 r - \partial_X r^3 + \varepsilon[-\tfrac{81}{2}\partial_X^2 r - \tfrac{1}{2}\partial_x^4 r + \tfrac{1}{2}\partial_X^2 r^3]. \tag{16}$$

If we ignore the O(ε) terms in equation (16), this is just the modified KdV equation with a kink solution as the desired solution:

$$r_0(X, T) = \sqrt{\frac{c}{2}}(X - cT). \tag{17}$$

Next, assuming that $r(X, T) = r_0(X, T) + \varepsilon r_1(X, T)$, we take into account the O(ε) correction. In order to determine the selected value of the propagation velocity c for the kink solution (17), it is necessary to satisfy the solvability condition [27]

$$(\Phi_0, M[r_0]) \equiv \int_{-\infty}^{+\infty} \mathrm{d}X\, \Phi_0 M[r_0] = 0 \tag{18}$$

where $\Phi_0 = r_0$ and $M[r_0] = \frac{81}{2}\partial_x^2 r + \frac{1}{2}\partial_X^4 r - \frac{1}{2}\partial_X^2 r^3$.

By performing the integration, we obtain the selected velocity $c = 81$. The amplitude of the kink solution is given by

$$\text{Amplitude} = \varepsilon C_1\sqrt{c} = 3\varepsilon. \tag{19}$$

Thus, we obtain the phase separation line in the $(\tau, \Delta x)$-plane. It is shown together with simulated values and the neutral stability curve in figure 1. The analytical result is in good agreement with the numerical simulation for small values of $\varepsilon^2 = \tau - \tau_c$ where $\tau_c = \frac{1}{3}$ and $V'(h_c) = 1$. This is a satisfying result since the nonlinear analysis only accounts for values close to the critical point (h_c, τ_c).

5. Simulation

We carry out simulation to derive numerically the phase separation line and to compare the simulation result with the analytical result. The investigated system is a difference–difference equation which can be referred back to a difference–differential model used by Newell [28] or to the differential equation model used by Bando *et al* [3]. Both models are able to simulate traffic patterns including the jamming transition. However, *a priori* it could not be assumed that the difference–difference equation model would yield similar patterns. Therefore simulation is carried out to validate two points. (1) First it has to be shown that the difference–difference equation model is indeed capable of describing traffic dynamics. (2) Next the applicability of the nonlinear analysis has to be proven.

For the first point spacetime diagrams are plotted at various densities and delay times using periodic boundary conditions and randomly chosen initial conditions. As a result, three types of traffic flow have to be distinguished: (1) a freely moving phase, (2) a coexisting phase in which jams appear and (3) a uniform congested phase. A pattern for the coexisting

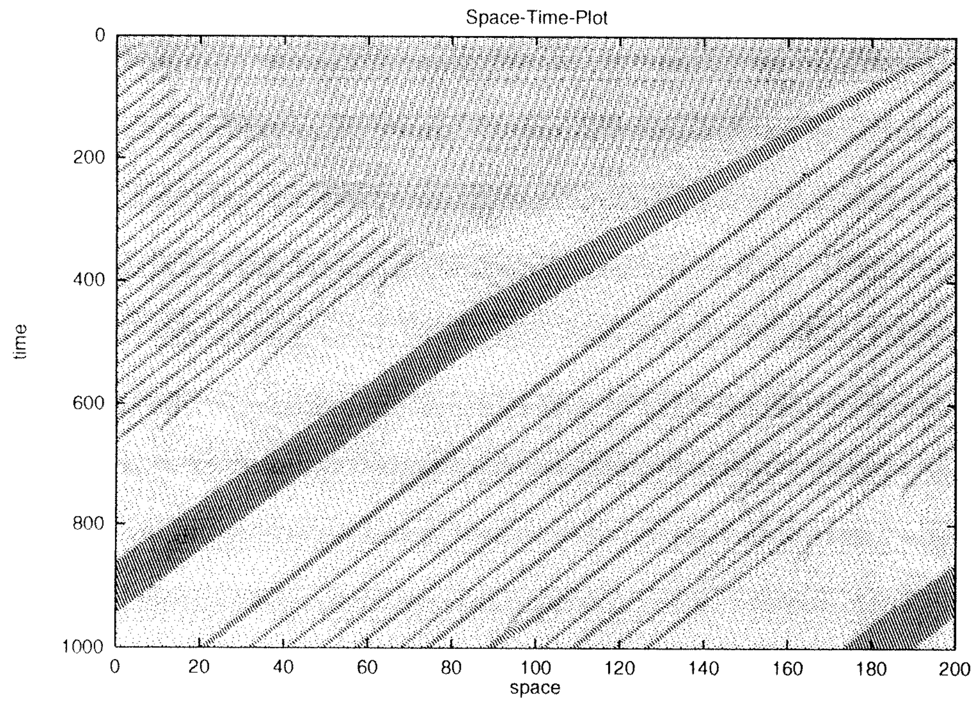

Figure 2. The spacetime plot for our model taken in the coexisting phase, thus backwards moving jams evolve. Each black dot represents a car, whereas white regions correspond to empty road space.

phase is given in figure 2. It shows the evolution of the system inside a region of 200 length units. The space coordinate is plotted on the x-axis, whereas the y-axis accounts for the time. 1000 timesteps are shown with time running from top to bottom. For this pattern initial conditions are chosen as follows

$$x_0(0) = x_{0,0} + 0.1 \qquad x_n(0) = x_{n,0} \text{ for } n \neq 0 \qquad \text{and} \qquad x_n(1) = x_n(0) + 0.9 \tag{20}$$

where $x_{1,0}$ accounts for a density-dependent homogeneous spacing and $h_c = 4$. In any case considering long-time evolution only two distinct densities survive for the coexisting phase depending on the delay time. They are the densities of the transition points. Thus for each delay time τ they are calculated from simulation giving rise to the points of the simulated curve in figure 1. Systems with a delay time below the critical point τ_c depicted in figure 1 differ from systems with delay times above τ_c by displaying no transition. This gives rise to a steady fundamental diagram, whereas there are two unsteady points characterizing the transitions for fundamental diagrams with larger delay times as shown in figure 3. For the fundamental diagram systems of 10^4 unit length are simulated and measurement steps are inserted after time $t = 5 \times 10^4$, where the systems are found to be in an equilibrium state.

6. Summary

We proposed the traffic model described by the difference equation and investigated the traffic's behaviour analytically by the use of the linear stability theory and the nonlinear wave analysis. We showed that there is a critical point below which no jams occur. A phase

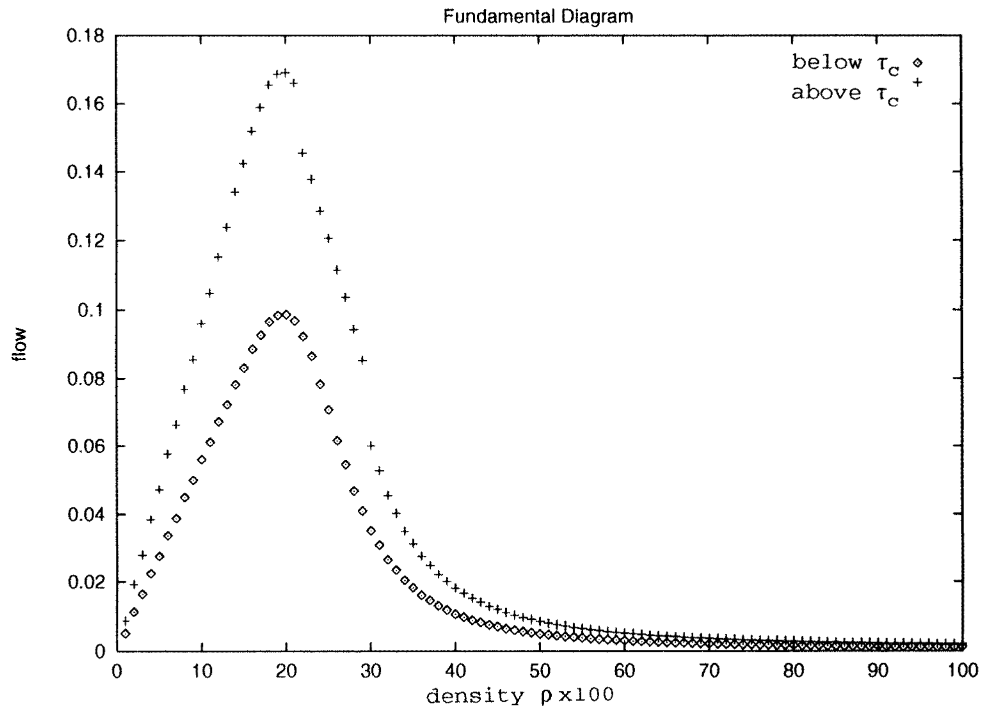

Figure 3. Fundamental diagram with the continuous curve below the critical point and discontinuities above. The discontinuities correspond to the phase transition.

transition occurs among the freely moving phase, the coexisting phase and the uniform congested phase. We derived the modified KdV equation to describe the traffic jam near the critical point. We also performed a numerical simulation for the difference equation model. We obtained the phase diagram and compared the analytical result with the simulation result. The analytical and simulation results near the critical point are in good agreement.

However, we would like to point out that a linear stability analysis as performed in this paper does not always yield meaningful results. As an example one may consider the following equation

$$x_j(t+\tau) = x_j(t) + V(\Delta x_j(t))\tau. \tag{21}$$

We note that there are no jams in the traffic flow described by equation (21), even though a neutral stability line can be derived analytically.

Appendix

In this appendix we give the derivation of a modified KdV equation (14) from the difference equation (11). We expand each term in equation (11) to the fifth-order of ε.

$$\begin{aligned}\Delta x_j(t+\tau) = h_c + C_1[\varepsilon r + \varepsilon^2 b\tau\partial_X r + \varepsilon^3((b\tau)^2/2)\partial_X^2 r + \varepsilon^4((b\tau)^3/6)\partial_X^3 r \\ +\varepsilon^5((b\tau)^4/24)\partial_X^4 r + \varepsilon^4 C_2\tau\partial_T r + \varepsilon^5 C_2 b\tau^2\partial_T\partial_X r].\end{aligned} \tag{A1}$$

Similarly, we obtain the expansion of $\Delta x_j(t+2\tau)$. The optimal velocity function is expanded as follows

$$V(\Delta x_j) = V(h_c) + V'(h_c)(\Delta x_j - h_c) + V'''(h_c)(\Delta x_j - h_c)^3/6 \tag{A2}$$

where $V''(h_c) = 0$ and $V'''(h_c) = 0$ in equation (4). The difference of the headways is expanded as follows

$$\Delta x_{j+1}(t) - \Delta x_j(t) = \varepsilon^2 C_1 \partial_X r + \varepsilon^3 (C/2) \partial_X^2 r + \varepsilon^4 (C_1/6) \partial_X^3 r + \varepsilon^5 (C_1/24) \partial_X^4 r. \qquad \text{(A3)}$$

The third power of headways is expanded as follows

$$(\Delta x_{j+1} - h_c)^3 - (\Delta x_j - h_c)^3 = \varepsilon^4 C_1^3 \partial_X r^3 + \varepsilon^5 (C_1^3/2) \partial_X^2 r^3. \qquad \text{(A4)}$$

By inserting these expansions (A1)–(A4) into equation (11), we obtain the nonlinear equation (14).

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